

Belokurov-Usyukina loop reduction in non-integer dimension

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Abstract

Belokurov-Usyukina loop reduction method has been proposed in 1983 to reduce a number of rungs in triangle ladder-like diagram by one. The disadvantage of the method is that it works in $d = 4$ dimensions only and it cannot be used for calculation of amplitudes in field theory in which we are required to put all the incoming and outgoing momenta on shell. We generalize the Belokurov-Usyukina loop reduction technique to non-integer $d = 4 - 2\varepsilon$ dimensions. In this paper we show how a two-loop triangle diagram with particular values of indices of scalar propagators in the position space can be reduced to a combination of three one-loop scalar diagrams. It is known that any one-loop massless momentum integral can be presented in terms of Appell's function F_4 . This means that particular diagram considered in the present paper can be represented in terms of Appell's function F_4 too. Such a generalization of Belokurov-Usyukina loop reduction technique allows us to calculate that diagram by this method exactly without decomposition in terms of the parameter ε .

Keywords: Loop reduction, Appell's function F_4 .

1 Introduction

The main part of the results of multi-loop calculus in high-energy physics has been done as an expansion in terms of ε that is the parameter of dimensional regularization [1, 2]. However, one-loop massless diagrams can be calculated in all order in ε and can be represented in terms of Appell's hypergeometric function [4, 3]. To calculate any loop (in momentum space) integral it would be good to have a technique that reduces number of loops by one in a recursive manner. Such a method exists in $d = 4$ space-time dimensions for triangle-ladder diagrams. It was discovered in early eighties by Belokurov and Usyukina [6, 5, 7]. We refer to that construction as to Belokurov-Usyukina loop reduction technique. As to our knowledge, there was no analog of this loop reduction procedure in $d = 4 - 2\varepsilon$ dimensions. The result of calculation of the triangle ladder diagrams in $d = 4$ is UD functions [8, 9, 10]. Their properties, in particular the invariance with respect to Fourier transform, have been studied in Refs. [11, 12, 13, 14] and their MB transforms have been studied in Refs. [14, 15].

In this paper we propose generalization of the Belokurov-Usyukina loop reduction technique to non-integer dimensions. In particular, we consider a two-loop triangle diagram in which the propagator indices in the position space are $1 - \varepsilon$ or 1. We use uniqueness method and method of integration by parts [16, 17, 18, 19]. The detailed step-by-step construction for $d = 4$ space-time dimensions is presented in Ref. [15]. Here we construct analogs of fig. (2) and fig. (3) of Ref. [15] with slightly modified indices of line in order to apply uniqueness technique to the case of triangle ladder diagram in non-integer number of dimensions.

2 Loop reduction in $d = 4 - 2\varepsilon$ dimensions

The result of the reduction is presented in three figures, fig.(2) is continuation of fig.(1), and fig.(3) is continuation of fig. (2). The transformations depicted in the diagrams are integration by parts, triangle-star and star-triangle relations (for review of these relations, see Ref. [19]). As we can see, the final result depicted in fig. (3) is a sum of one-loop diagrams. Each of the diagrams on the r.h.s. of fig.(3) can be transformed to the momentum space in which the result for each one of them is a combination of Appell's functions [4]. To our knowledge, it is the first known case when two-loop diagram in non-integer number of dimensions can be reduced to the Appell's function F_4 in all order in the regularization parameter ε for arbitrary kinematic region in the momentum space.

The figures are self-explaining. A new d -dimensional measure $Dx \equiv \pi^{-\frac{d}{2}} d^d x$ introduced in ref. [20] is assumed in the position space to avoid powers of π in figures. The factor J that appears in figures (1)-(3) is

$$J = \frac{\Gamma(1 - \varepsilon_1)\Gamma(1 - \varepsilon_2)\Gamma(1 - \varepsilon_3)}{\Gamma(1 + \varepsilon_1 - \varepsilon)\Gamma(1 + \varepsilon_2 - \varepsilon)\Gamma(1 + \varepsilon_3 - \varepsilon)}.$$

This generalizes the corresponding factor J of Ref. [15]. The condition for auxiliary parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ remains the same as in Ref. [5, 7, 15],

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0.$$

At the end of the calculation we have to take the limit of vanishing these ε -terms.

3 Conclusion

We have shown that the loop reduction in non-integer number of dimensions apparently exists. The two-loop diagram in $d = 4 - 2\varepsilon$ dimensions has been represented as one-loop diagrams in the same kinematic region in the momentum space. We have considered an arbitrary kinematic region and even on-shell external momenta can be taken. In that case the result remains finite and regularized dimensionally in terms of poles in regularization parameter ε . However, not all of the indices in the position space are $1 - \varepsilon$. This index in the position space means index 1 in the momentum space, which corresponds to the physical case of momentum propagator in the regularized $(4 - 2\varepsilon)$ -dimensional theory.

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$$\begin{aligned}
& \text{Two-loop diagram} = -\frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{One-loop diagram} \\
& -\frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{One-loop diagram} + \frac{\Gamma(1-\varepsilon_2)\Gamma(-\varepsilon_3)\Gamma(-\varepsilon_1)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{One-loop diagram} \\
& = -\frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(-\varepsilon_3)\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_1)\Gamma(2+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_2-\varepsilon)} \text{One-loop diagram} \\
& -\frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \frac{\Gamma(1-\varepsilon_3)\Gamma(2-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(\varepsilon_2)\Gamma(1+\varepsilon_1)} \text{One-loop diagram} \\
& +\frac{\Gamma(1-\varepsilon_2)\Gamma(-\varepsilon_3)\Gamma(-\varepsilon_1)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{One-loop diagram} = \frac{J}{\varepsilon_2\varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)} \frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)} \text{One-loop diagram} \\
& +\frac{J}{\varepsilon_1\varepsilon_2} \Gamma^2(1-\varepsilon) \text{One-loop diagram} +\frac{J}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)} \frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)} \text{One-loop diagram} \\
& = \frac{J}{\varepsilon_2\varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)} \frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)} \text{One-loop diagram} +\frac{J}{\varepsilon_1\varepsilon_2} \Gamma^2(1-\varepsilon) \text{One-loop diagram} \\
& +\frac{J}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)} \frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)} \text{One-loop diagram}
\end{aligned}$$

Figure 1. Reduction of two-loop diagram

$$\begin{aligned}
&= \frac{J}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)} \frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)} \frac{\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1)\Gamma(-\varepsilon_3)} \frac{2+\varepsilon_3-\varepsilon}{1+\varepsilon_1-\varepsilon} \frac{1-\varepsilon_3}{1+\varepsilon_2} \frac{1+\varepsilon_1}{1+\varepsilon_3} \\
&+ \frac{J}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)} \frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)} \frac{\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1)\Gamma(-\varepsilon_3)} \frac{2+\varepsilon_3-\varepsilon}{1-\varepsilon} \frac{1-\varepsilon_3}{1+\varepsilon_1-\varepsilon} \frac{1+\varepsilon_2}{1+\varepsilon_3} \\
&+ \frac{J}{\varepsilon_1 \varepsilon_2} \Gamma^2(1-\varepsilon) \frac{2+\varepsilon_3-\varepsilon}{1-\varepsilon} \frac{1-\varepsilon_3}{1+\varepsilon_2} \frac{1+\varepsilon_1}{1+\varepsilon_3} = \frac{J}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)} \frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)} \frac{1}{2+\varepsilon_3-\varepsilon} \frac{1-\varepsilon}{1+\varepsilon_1-\varepsilon} \frac{1+\varepsilon_1}{1+\varepsilon_3} \\
&- \frac{J}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon)}{1+\varepsilon_3-\varepsilon} \frac{1}{1-\varepsilon} \frac{1+\varepsilon_1}{1+\varepsilon_3} \frac{1+\varepsilon_2}{1+\varepsilon_2} + \frac{J}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)} \frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)} \frac{1}{2+\varepsilon_3-\varepsilon} \frac{1-\varepsilon}{1+\varepsilon_1-\varepsilon} \frac{1+\varepsilon_2}{1+\varepsilon_3} \\
&= \frac{J}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)} \frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)} \frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_3-\varepsilon)\Gamma(2-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_3)\Gamma(\varepsilon_2)} \frac{1}{2+\varepsilon_3-\varepsilon} \frac{1-\varepsilon}{1-\varepsilon_1} \frac{1+\varepsilon_1}{1+\varepsilon_2} \\
&- \frac{J}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon)}{1+\varepsilon_3-\varepsilon} \frac{1}{1-\varepsilon} \frac{1+\varepsilon_1}{1+\varepsilon_3} \frac{1+\varepsilon_2}{1+\varepsilon_2} + \frac{J}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)} \frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)} \frac{\Gamma(1-\varepsilon_2)\Gamma(1-\varepsilon_3-\varepsilon)\Gamma(2-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_2-\varepsilon)\Gamma(1+\varepsilon_3)\Gamma(\varepsilon_1)} \frac{1}{2+\varepsilon_3-\varepsilon} \frac{1-\varepsilon}{1-\varepsilon_2} \frac{1+\varepsilon_1}{1+\varepsilon_2}
\end{aligned}$$

Figure 2. Reduction of two-loop diagram. Continuation of fig. (1)

$$\begin{aligned}
&= -\frac{J}{\varepsilon_3(1+\varepsilon_3-\varepsilon)} \left[\frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)} \frac{1}{1-\varepsilon} \frac{1+\varepsilon_1}{1-\varepsilon_1} \frac{1+\varepsilon_2}{1+\varepsilon_2} \right. \\
&\quad \left. + \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)} \frac{1}{1-\varepsilon} \frac{1+\varepsilon_1}{1-\varepsilon_2} \frac{1+\varepsilon_2}{1+\varepsilon_2} \right]
\end{aligned}$$

Figure 3. Final result

References

- [1] V. A. Smirnov, “Evaluating Feynman Integrals,” Springer Tracts in Modern Physics **211** (2004), Springer, Germany
- [2] Z. Bern, L. J. Dixon and V. A. Smirnov, “Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond,” Phys. Rev. D **72** (2005) 085001 [hep-th/0505205].
- [3] E. E. Boos and A. I. Davydychev, “A Method of evaluating massive Feynman integrals,” Theor. Math. Phys. **89** (1991) 1052 [Teor. Mat. Fiz. **89** (1991) 56].
- [4] A. I. Davydychev, “Recursive algorithm of evaluating vertex type Feynman integrals,” J. Phys. A **25**, 5587 (1992).
- [5] V. V. Belokurov and N. I. Usyukina, “Calculation Of Ladder Diagrams In Arbitrary Order,” J. Phys. A **16** (1983) 2811.
- [6] N. I. Usyukina, “Calculation Of Many Loop Diagrams Of Perturbation Theory,” Theor. Math. Phys. **54** (1983) 78 [Teor. Mat. Fiz. **54** (1983) 124].
- [7] N. I. Usyukina, “Calculation of multiloop diagrams in arbitrary order,” Phys. Lett. B **267** (1991) 382 [Theor. Math. Phys. **87** (1991) 627] [Teor. Mat. Fiz. **87** (1991) 414]
- [8] N. I. Usyukina and A. I. Davydychev, “An Approach to the evaluation of three and four point ladder diagrams,” Phys. Lett. B **298** (1993) 363.
- [9] N. I. Usyukina and A. I. Davydychev, “Exact results for three and four point ladder diagrams with an arbitrary number of rungs,” Phys. Lett. B **305** (1993) 136.
- [10] D. J. Broadhurst and A. I. Davydychev, “Exponential suppression with four legs and an infinity of loops,” Nucl. Phys. Proc. Suppl. **205-206** (2010) 326 [arXiv:1007.0237 [hep-th]].
- [11] I. Kondrashuk and A. Kotikov, “Fourier transforms of UD integrals,” arXiv:0802.3468 [hep-th], Birkhauser book series “Trends in Mathematics”, volume “Analysis and Mathematical Physics”, B. Gustafsson and A. Vasil’ev (Eds), (2009) Birkhauser Verlag, Basel, Switzerland, 337-348
- [12] I. Kondrashuk and A. Kotikov, “Triangle UD integrals in the position space,” JHEP **0808** (2008) 106 [arXiv:0803.3420 [hep-th]].
- [13] I. Kondrashuk and A. Vergara, “Transformations of triangle ladder diagrams,” JHEP **1003** (2010) 051 [arXiv:0911.1979 [hep-th]].
- [14] P. Allendes, N. Guerrero, I. Kondrashuk and E. A. Notte Cuello, “New four-dimensional integrals by Mellin-Barnes transform,” J. Math. Phys. **51** (2010) 052304 [arXiv:0910.4805 [hep-th]].
- [15] P. Allendes, B. Kniehl, I. Kondrashuk, E. A. Notte Cuello and M. Rojas Medar, “Solution to Bethe-Salpeter equation via Mellin-Barnes transform,” arXiv:1205.6257 [hep-th].
- [16] M. D’Eramo, L. Peliti and G. Parisi, “Theoretical Predictions for Critical Exponents at the λ -Point of Bose Liquids,” Lett. Nuovo Cimento **2** (1971) 878.

- [17] A. N. Vasiliev, Y. M. Pismak and Y. R. Khonkonen, “ $1/N$ Expansion: Calculation Of The Exponents η And ν In The Order $1/N^2$ For Arbitrary Number Of Dimensions,” *Theor. Math. Phys.* **47** (1981) 465 [*Teor. Mat. Fiz.* **47** (1981) 291].
- [18] A.N. Vasiliev, “The field theoretic renormalization group in critical behaviour theory and stochastic dynamics”, St. Petersburg Institute of Nuclear Physics Press, 1998.
- [19] D. I. Kazakov, “Analytical Methods For Multiloop Calculations: Two Lectures On The Method Of Uniqueness,” JINR-E2-84-410.
- [20] G. Cvetič, I. Kondrashuk, A. Kotikov and I. Schmidt, “Towards the two-loop L_{cc} vertex in Landau gauge,” *Int. J. Mod. Phys. A* **22** (2007) 1905 [hep-th/0604112].
- [21] D. Binosi, L. Theussl, “JaxoDraw: A graphical user interface for drawing Feynman diagrams” *Comp. Physics Comm.* **161** (2004) 76